ISSN 1392 - 1207. MECHANIKA. 2007. Nr.5(67)

Optimal shakedown design of plates

J. Atkočiūnas*, L. Rimkus, V. Skaržauskas***, E. Jarmolajeva******
**Vilnius Gediminas Technical University, Saulėtekio al. 11, 10223 Vilnius, Lithuania,*
E-mail: juozas.atkociunas@st.vgtu.lt
***Vilnius Gediminas Technical University, Saulėtekio al. 11, 10223 Vilnius, Lithuania,*
E-mail: liudvikas.rimkus@adm.vgtu.lt
****Vilnius Gediminas Technical University, Saulėtekio al. 11, 10223 Vilnius, Lithuania,*
E-mail: valentinas.skarzauskas@adm.vgtu.lt,
*****Vilnius Gediminas Technical University, Saulėtekio al. 11, 10223 Vilnius, Lithuania,*
E-mail: ela.chraptovic@st.vgtu.lt

1. Problem formulation

Elastic-plastic steel bending plate subjected to a repeated variable load (RVL) is considered in this paper. RVL is the system of loads the each of which can independently vary within the prescribed bounds. Ideal elastic-plastic construction subjected by afore mentioned load can lose its exploitative suitability due to failure caused by progressive plastic and/or alternating straining. Usually both cases are denoted as cyclic plastic collapse. Note that structure can adapt to repeated variable load and subsequently response to RVL in elastic range. Shakedown analysis via numerical and mathematical programming methods of elastic-plastic any complexity structure, subjected to RVL is relevant for civil engineering. This is confirmed by the growing number of investigations in this field [1]. However one can find only several works concerning optimization of adapted structures. Therefore current investigation is actual.

The solution of structure optimization problem at shakedown is complicated because stress-strain state of dissipative system (e.g. the plate plastic deforming) depends on loading history [1-12]. The optimization problem is stated by involving extreme energy principles and methods of mathematical programming theory. New iterative algorithm of problem approximate solution for adapted flexural plates optimization based on Rosen project gradient method [13] is proposed in this paper. A mathematical model in static formulation is constructed to determine shakedown stress-strain state of flexural plates. The dual problem solution (kinematical formulation of the problem) is obtained by applying mathematical-mechanical interpretation of Rosen criterion. This methodology previously was explained by authors in [14].

The problem of determining optimal distribution of plate parameters at cyclic-plastic collapse is considered as a separate case of optimization at shakedown state. The relationship between afore mentioned mathematical models and iterative Rosen algorithm is employed to develop an approximate method for the solution of optimization problems.

2. Plate analysis problem

2.1. Plate discrete model, main equations and relationships

A discrete model is derived dividing the plate into s finite elements, every of which contains s_k nodal points [15, 16]. Thus the total sections number of plate discrete model is $\zeta = s \times s_k$. So, yield conditions will be verified in aforementioned nodal points. Stress-strain field of discrete model is described by n -size vectors of bending moments and strains $\boldsymbol{M} = (\boldsymbol{M}_1, \boldsymbol{M}_2, ..., \boldsymbol{M}_\zeta)^T$ $\boldsymbol{\Theta} = (\boldsymbol{\Theta}_1, \boldsymbol{\Theta}_2, ..., \boldsymbol{\Theta}_\zeta)^T$, respectively.

Let the degree of freedom of the plate equilibrium finite element to be denoted via m. Then equilibrium equations taking into account boundary conditions presented in the general form are

$$\sum_k \boldsymbol{A}_k \boldsymbol{M}_k = \boldsymbol{F} \text{ or } \boldsymbol{A}\boldsymbol{M} = \boldsymbol{F} \quad (1)$$
$$k = 1,2,...,s; \quad k \in K$$

here the size of the matrix of equilibrium equations coefficients $\boldsymbol{A}$ is ($m \times n$), where n is total number of vector components vector of internal forces $\boldsymbol{M}$.

Geometrical equations for separate finite element read

$$\boldsymbol{A}_k^T \boldsymbol{u} - \boldsymbol{D}_k \boldsymbol{M}_k = \boldsymbol{0}, \quad k \in K \quad (2)$$

then for the whole discrete system one obtains

$$\boldsymbol{A}^T \boldsymbol{u} - \boldsymbol{D}\boldsymbol{M} = \boldsymbol{0} \quad (3)$$

here $\boldsymbol{D}$ is $n \times n$ size matrix of elemental flexibilities $\boldsymbol{D}_k$ of the plate discrete model; $\boldsymbol{u} = (\boldsymbol{u}_1, \boldsymbol{u}_2, ..., \boldsymbol{u}_m)^T$ is displacement vector.

Huber-Mizes nonlinear yield condition reads: for rectangular plate $M_{11}^2 - M_{11}M_{22} + M_{22}^2 + 3M_{12}^2 \le M_0^2$; for circular plate $M_\rho^2 - M_\rho M_\theta + M_\theta^2 \le M_0^2$. Yield conditions are verified at all design sections of the plate (i.e. at every element node)

$$\varphi_{kl} = C_k - \boldsymbol{M}_{kl}^T \boldsymbol{\Pi}_{kl} \boldsymbol{M}_{kl} \ge 0, \quad C_k = (M_{0k})^2 \quad (4)$$
$$k = 1,2,...,s; \quad l = 1,2,..., s_k$$

here M_{0k} is limit bending moment assumed to be constant per finite element area. Steel plate of continuous cross-section is analyzed. Then $M_0 = \frac{1}{4}\sigma_y h^2$, where h denotes plate thickness, σ_y denotes yield limit. Matrix of coefficients of the yield condition (4) for rectangular plate reads

$\boldsymbol{\Pi}_{kl} =$

1	- 0.5	0
- 0.5	1	0
0	0	3

here (M_{11}, M_{22} denote bending, M_{12} denote torsion). As only radial M_ρ and circular M_θ moments describe the stress state of circular plate, the matrix $\boldsymbol{\Pi}_{kl}$ is simplified

$\boldsymbol{\Pi}_{kl} =$

1	-0.5
-0.5	1

Mostly the variable repeated load $\boldsymbol{F}(t)$ is defined not via particular loading history but only by constant bounds $\boldsymbol{F}_{sup}$, $\boldsymbol{F}_{inf}$ of the upper and lower load variation. Then $\boldsymbol{F}_{inf} \le \boldsymbol{F}(t) \le \boldsymbol{F}_{sup}$. Optimization problems of adapted plate is solved accounting only the load variation bounds. Then the structure, undergoing plastic strains $\boldsymbol{\Theta}_p$ in the early loading cycles, further adapts to the load. Residual bending moments $\boldsymbol{M}_r$ conditioned by plastic strains $\boldsymbol{\Theta}_p$ ensure that the subsequent load variation does not cause the development of other plastic strains. Here subscript e denotes variables of elastic response, subscript r denote residual internal forces, strains and deflections.

Then employing the aforementioned definitions the vector of total moments $\boldsymbol{M}_{kl}$ (see yield conditions (4)) and taking into account elastic moments $\boldsymbol{M}_{ekl}(t)$) reads

$$\boldsymbol{M}_{kl}(t) = \boldsymbol{M}_{ekl}(t) + \boldsymbol{M}_{rkl}, \; k = 1,2,...,s; \; l = 1,2,...,s_k \quad (5)$$

here $\boldsymbol{M}_{ekl}(t) = \boldsymbol{\alpha}\boldsymbol{F}(t)$ is valid for the whole plate discrete model, $\boldsymbol{\alpha}$ is the matrix of elastic response under bending moment influence. The m-size volume of variation of elastic internal forces $\boldsymbol{M}_{ekl}(t)$ taking into account possible combinations (the total number of them $p = 2^m$) of loads $\boldsymbol{F}_{inf}$, $\boldsymbol{F}_{sup}$ is bounded by the convex and symmetric polyhedron. Denote the apexes of polyhedron via $\boldsymbol{M}_{ej}$, $j = 1, 2,..., p$; $j \in J$. Omitting the detailed investigation of loading history the yield conditions finally take the form

$$\varphi_{kl,j} = C_k - \boldsymbol{M}_{kl,j}^T \boldsymbol{\Pi}_{kl} \boldsymbol{M}_{kl,j} \ge 0 \quad (6)$$
$$\boldsymbol{M}_{kl,j} = \boldsymbol{M}_{ekl,j} + \boldsymbol{M}_{rkl}$$
$$k = 1,2,...,s; \quad l = 1,2,..., s_k; \quad j = 1, 2, ..., p$$

It is convenient to pick out the residual bending moments $\boldsymbol{M}_r$, the displacements $\boldsymbol{u}_r$ and the strains $\boldsymbol{\Theta}_r = \boldsymbol{D}\boldsymbol{M}_r + \boldsymbol{\Theta}_p$ when analyzing the structure at shakedown. The equilibrium (1) and geometrical (3) equations in this case read

$$\sum_k \boldsymbol{A}_k \boldsymbol{M}_{rk} = \boldsymbol{0} \text{ or } \boldsymbol{A}\boldsymbol{M}_r = \boldsymbol{0} \quad (7)$$

and

$$\boldsymbol{A}^T \boldsymbol{u}_r = \boldsymbol{D}\boldsymbol{M}_r + \boldsymbol{\Theta}_p \quad (8)$$

here the components of the vector of plastic strains $\boldsymbol{\Theta}_p = (\boldsymbol{\Theta}_{pkl})^T$ are obtained by

$$\boldsymbol{\Theta}_{pkl} = \sum_j \left[\nabla\varphi_{kl,j}\left(\boldsymbol{M}_{ekl,j} + \boldsymbol{M}_{rkl}\right)\right]^T \lambda_{kl,j} \quad (9)$$
$$\lambda_{kl,j} \ge 0$$
$$k = 1,2,...,s; \quad l = 1,2,..., s_k; \quad j = 1, 2, ..., p$$

here $\lambda_{kl,j}$ is the plastic multiplier, $\nabla\boldsymbol{\varphi}$ is the gradient matrix of yield conditions [17].

2.2. Mathematical model of analysis problem

The calculation of residual internal forces and strains of adapted plate for given RVL $\boldsymbol{F}_{inf} \le \boldsymbol{F}(t) \le \boldsymbol{F}_{sup}$ is analyzed in the section. The plate parameters, including the limit bending moments $\boldsymbol{M}_{0k}$ ($k \in K$), are prescribed values.

The problem in static formulation represents the minimum complementary energy principle reading: of all statically admissible vectors of residual bending moments $\boldsymbol{M}_r$ at shakedown is the minimum complementary energy corresponding one. The problem mathematical, model stated on the basis of above-mentioned principle, reads find

$$min \; \frac{1}{2}\sum_k \boldsymbol{M}_{rk}^T \boldsymbol{D}_k \boldsymbol{M}_{rk} = a^* \quad (10)$$

subject to

$$\sum_k [A_k] \boldsymbol{M}_{rk} = \boldsymbol{0} \quad (11)$$

$$\varphi_{kl,j} = C_k - \left(\boldsymbol{M}_{ekl,j} + \boldsymbol{M}_{rkl}\right)^T \boldsymbol{\Pi}_{kl} \left(\boldsymbol{M}_{ekl,j} + \boldsymbol{M}_{rkl}\right) \ge \boldsymbol{0} \quad (12)$$

$$C_k = \left(M_{0k}\right)^2$$
$$k = 1,2,...,s; \quad l = 1,2,..., s_k, \quad j = 1, 2, ..., p \quad (13)$$

Conditions (11)–(13) define a field of convex admissible solutions of the problem (10)–(13). Plate bending limit moments M_{0k} (M_{0k} is considered to be constant in the finite element area) and bending moments of elastic response $\boldsymbol{M}_{ekl,j}$ are prescribed (known) values in the convex mathematical programming problem (10)–(13).

The optimal solution $\boldsymbol{M}_r^*$ of the problem (10)–(13) is unique, i.e. the aforementioned residual bending moments ensure the plate adaptation to the fixed RVL $\boldsymbol{F}_{inf} \le \boldsymbol{F}(t) \le \boldsymbol{F}_{sup}$. The yield conditions for optimal solution $\boldsymbol{M}_r^*$, satisfied as equalities, are denoted to be the active conditions.

The dual problem to the problem (10)–(13) reads find

$$max \left\{ -\frac{1}{2}\sum_k \boldsymbol{M}_{rk}^T \boldsymbol{D}_k \boldsymbol{M}_{rk} - \sum_j \boldsymbol{\lambda}_j \left[\nabla\boldsymbol{\varphi}_j \left(\boldsymbol{M}_{ej} + \boldsymbol{M}_r\right)\right] \boldsymbol{M}_r - \right.$$

$$-\sum_j \boldsymbol{\lambda}_j^T \left[\boldsymbol{C} - \boldsymbol{f}_j\left(\boldsymbol{M}_{ej} + \boldsymbol{M}_r\right) \right] \Bigg\} \tag{14}$$

subject to

$$\boldsymbol{D}\boldsymbol{M}_r + \sum_j \left[\nabla\boldsymbol{\varphi}_j \left(\boldsymbol{M}_{ej} + \boldsymbol{M}_r\right) \right]^T \boldsymbol{\lambda}_j - \boldsymbol{A}^T \boldsymbol{u}_r = \boldsymbol{0} \tag{15}$$

$$\boldsymbol{\lambda}_j \geq \boldsymbol{0}, \quad k \in K, \quad j \in J \tag{16}$$

here

$$\boldsymbol{f}_j\left(\boldsymbol{M}_{ej} + \boldsymbol{M}_r\right) = \left[\boldsymbol{\Gamma}\left(\boldsymbol{M}_{ej} + \boldsymbol{M}_r\right)\right] \boldsymbol{\Pi}\left(\boldsymbol{M}_{ej} + \boldsymbol{M}_r\right) \tag{17}$$

$\boldsymbol{\Gamma}$, $\boldsymbol{\Pi}$ are the quasi-diagonal matrices.

The problem (14)–(16) corresponds the following energy principle: *of all kinematically admissible residual displacements the vector* $\boldsymbol{u}_r$ *at shakedown is the minimum total potential energy corresponding one*.

The optimal solution of problem (14)–(16) are the vectors $\boldsymbol{M}_r^*$, $\boldsymbol{u}_r^*$ and $\boldsymbol{\lambda}_j^*$. The maximum value of dissipated energy at shakedown is expressed by

$$D_{max} = \sum_j \boldsymbol{\lambda}_j^T \boldsymbol{M}_0 \tag{18}$$

The plate residual strains $\boldsymbol{\Theta}_r^* = \boldsymbol{D}\boldsymbol{M}_r^* + \boldsymbol{\Theta}_p^*$ (strains $\boldsymbol{\Theta}_p^*$ are calculated by formula (9)) and residual displacements $\boldsymbol{u}_r^*$ at shakedown can be nonunique: they depend on certain loading history $\boldsymbol{F}(t)$. Thus, if the structural load is described only by it's variation bounds $\boldsymbol{F}_{inf}$, $\boldsymbol{F}_{sup}$ the identification of exact residual displacements values becomes problematic. This is conditioned by non-monotonic variation nature of aforementioned values at shakedown process.

2.3. Rosen algorithm and dual solution of analysis problem

The mathematical and mechanical sense of Rosen design gradient algorithm optimality criterion, reading

$$\left\{ \boldsymbol{I} - \nabla^T\boldsymbol{\varphi} \left(\nabla\boldsymbol{\varphi}\, \nabla^T\boldsymbol{\varphi} \right)^{-1} \nabla\boldsymbol{\varphi} \right\} \nabla\mathscr{F} = \boldsymbol{0} \tag{19}$$

$$\left(\nabla\boldsymbol{\varphi}\nabla^T\boldsymbol{\varphi} \right)^{-1} \nabla\boldsymbol{\varphi}\nabla\mathscr{F} \geq \boldsymbol{0} \tag{20}$$

was explained in investigation [14]. In (19)–(20) the relation $\nabla\mathscr{F}$ denote the gradient of the objective function (10). The equations (19) represent the compatibility equations of residual strains. The relations (20) represent the vector of plastic multipliers $\boldsymbol{\lambda}$ which is related with active conditions of problem (10)–(13).

On the other hand the paper [14] proved that Rosen optimality criterion corresponds the Kuhn-Tucker conditions for the minimization problem (10)–(13). Thus, when solving analysis problem in static formulation (10)–(13) for adapted plate one simultaneously obtains the optimal solution $\boldsymbol{M}_r^*$ of the primal problem and the optimal solution $\boldsymbol{M}_r^*$, $\boldsymbol{u}_r^*$, $\boldsymbol{\lambda}_j^*$ ($j \in J$) of the dual problem (14)–(16).

2.4. Residual displacements and influence matrices for bending moments

As the plastic strains $\boldsymbol{\Theta}_p$ are known one can calculate the residual displacements $\boldsymbol{u}_r$ and the bending moments $\boldsymbol{M}_r$. Here the influence matrices $\boldsymbol{H}$ and $\boldsymbol{G}$ are introduced [18, 19]

$$\begin{aligned} \boldsymbol{u}_r &= \left(\boldsymbol{A}\boldsymbol{D}^{-1}\boldsymbol{A}^T \right)^{-1} \boldsymbol{A}\boldsymbol{D}\boldsymbol{\Theta}_p = \boldsymbol{\beta}\boldsymbol{A}\boldsymbol{D}^{-1}\boldsymbol{\Theta}_p = \\ &= \boldsymbol{\alpha}^T\boldsymbol{\Theta}_p = \boldsymbol{H}\boldsymbol{\Theta}_p \end{aligned} \tag{21}$$

$$\boldsymbol{M}_r = \left(\boldsymbol{D}^{-1}\boldsymbol{A}^T\boldsymbol{\beta}\boldsymbol{A}\boldsymbol{D}^{-1} - \boldsymbol{D}^{-1} \right)\boldsymbol{\Theta}_p = \boldsymbol{G}\boldsymbol{\Theta}_p \tag{22}$$

where $\boldsymbol{\beta}$ is displacements influence matrix of the plate elastic response. Matrix $\boldsymbol{G}$ is singular, i.e. it's inverse matrix does not exist.

Total displacements and internal forces of the plate subjected by RVL are calculated applying the formulae

$$\boldsymbol{u}(t) = \boldsymbol{u}_e(t) + \boldsymbol{u}_r = \boldsymbol{\beta}\boldsymbol{F}(t) + \boldsymbol{H}\boldsymbol{\Theta}_p \tag{23}$$

$$\boldsymbol{M}(t) = \boldsymbol{M}_e(t) + \boldsymbol{M}_r = \boldsymbol{\alpha}\boldsymbol{F}(t) + \boldsymbol{G}\boldsymbol{\Theta}_p \tag{24}$$

Extreme elastic displacements $\boldsymbol{u}_{e,\,sup}$, $\boldsymbol{u}_{e,\,inf}$ can be calculated applying the formulae

$$\boldsymbol{u}_{e,\,sup} = \boldsymbol{\beta}_{sup}\boldsymbol{F}_{sup} + \boldsymbol{\beta}_{inf}\boldsymbol{F}_{inf} \tag{25}$$

$$\boldsymbol{u}_{e,\,inf} = \boldsymbol{\beta}_{sup}\boldsymbol{F}_{inf} + \boldsymbol{\beta}_{inf}\boldsymbol{F}_{sup} \tag{26}$$

The components of the matrix $\boldsymbol{\beta}_{sup}$ are the elements $\beta_{ij} \geq 0$ of the matrix $\boldsymbol{\beta}$. Note that the components of the matrix $\boldsymbol{\beta}_{inf}$ satisfy conditions $\beta_{ij} \leq 0$ ($\boldsymbol{\beta} = \boldsymbol{\beta}_{sup} + \boldsymbol{\beta}_{inf}$).

Total extreme displacements then read

$$\boldsymbol{u}_{sup} = \boldsymbol{u}_{e,\,sup} + \boldsymbol{u}_{r,\,sup} \tag{27}$$

$$\boldsymbol{u}_{inf} = \boldsymbol{u}_{e,\,inf} + \boldsymbol{u}_{r,\,inf} \tag{28}$$

Taking into account that residual displacements $\boldsymbol{u}_r(t)$ in the loading process can vary non-monotonously one obtains

$$\boldsymbol{u}_{r,\,inf} \leq \boldsymbol{u}_r(t) \leq \boldsymbol{u}_{r,\,sup} \tag{29}$$

Creation of residual displacements vectors $\boldsymbol{u}_{r,\,inf}$, $\boldsymbol{u}_{r,\,sup}$ is exhaustively explained in the [11, 17]. Plate total bending moments are calculated applying the formula $\boldsymbol{M}_j = \boldsymbol{M}_{e,j} + \boldsymbol{M}_r$, which was already employed in the mathematical models (10)–(13), (14)–(16).

3. Optimal bending plate project: cyclic-plastic collapse

An optimal bending plate project is to be found for prescribed RVL and plate geometry. Let us assume that the price of the plate material volume, summarized from the area unit of the middle plane of the plate and the limit bending moment M_0 are directly proportional values. Then the theoretical plate price is

$$\omega(M_0)=\sum_k \varphi_k \overline{A}_k M_{0k} = \boldsymbol{L}^T \boldsymbol{M}_0 \quad (30)$$

here φ_k is scalar function of the limit bending moment unit, $\overline{A}_k$ is the area of the k-th finite element at middle surface, $\boldsymbol{L}=(L_1, L_2,..., L_s)^T$ is the vector of weight ratios of the optimality criterion. For the homogeneous plate $\varphi = const$. In our calculations it was taken the $\varphi_k = 1$. The components of the vector $\boldsymbol{L}$ become proportional to the areas of plate discrete model elements. Then, estimating the optimality criterion the expression (30) can be rewritten by

$$min\ \ \omega(M_0)= min\ \ \boldsymbol{L}^T \boldsymbol{M}_0 \quad (31)$$

The admissible and sufficient construction of optimality criterion condition is the constancy of the energy dissipation velocity $\dot{D}$ per unit volume of the construction (basing on Prager and Shield's work [20]). Then

$$\frac{\dot{D}}{\sum_k \varphi_k \overline{A}_k M_{0k}} = \alpha = const\ ,\ \ k=1,2,...,s \quad (32)$$

from here

$$\dot{D} = \alpha \sum_k \varphi_k \overline{A}_k M_{0k} = \alpha\, \boldsymbol{L}^T \boldsymbol{M}_0 = \dot{\boldsymbol{\Lambda}}^T \boldsymbol{M}_0 \quad (33)$$

where $\dot{\boldsymbol{\Lambda}}$ is intensities vector of plastic strains velocities. The minimal value of the linear function (33), physically meaning the energy dissipation rate, is reached on the edge of admissible solutions field. Project problem formulation is based on the principle of cyclic plastic collapse reading: *of all statically admissible residual bending moments* $\boldsymbol{M}_r$ *at cyclic plastic collapse the actual is the one corresponding to the minimum cycle energy dissipation rate* $\dot{D} = \dot{\boldsymbol{\Lambda}}^T \boldsymbol{M}_0$ [21]. The optimization problem mathematical model following the above-mentioned principle reads
find

$$min\ \sum_k \dot{\boldsymbol{\Lambda}}_k^T \boldsymbol{M}_{0k} = min\ \sum_k \boldsymbol{L}_k \boldsymbol{M}_{0k} \quad (34)$$

subject to

$$\sum_k \boldsymbol{A}_k \boldsymbol{M}_{rk} = \boldsymbol{0} \quad (35)$$

$$\varphi_{kl,j} = (M_{0k})^2 - (\boldsymbol{M}_{ekl,j} + \boldsymbol{M}_{rkl})^T \boldsymbol{\Pi}_{kl} (\boldsymbol{M}_{ekl,j} + \boldsymbol{M}_{rkl}) \geq 0 \quad (36)$$

$$k=1,2,...,s\ ;\ \ l=1, 2,..., s_k\ ;\ \ j=1, 2,..., p$$

Unknowns of the problem (34)–(36) are the vectors of limit $\boldsymbol{M}_0 = (M_{01},\ M_{02},...,\ M_{0s})^T$ and residual $\boldsymbol{M}_r$ bending moments. Optimal solution further is denoted via $\boldsymbol{M}_0^*$ and $\boldsymbol{M}_r^*$. Cyclic-plastic collapse corresponds to progressive or alternating plastic failures. Actual failure case can be defined by analyzing the solution of the optimization problem in kinematical formulation (due to plastic multipliers vector $\dot{\boldsymbol{\lambda}}_j$, $j \in J$). Bending moments at the cyclic-plastic collapse may not satisfy the criterion (10), i.e. $\frac{1}{2}\boldsymbol{M}_r^* \boldsymbol{D}\, \boldsymbol{M}_r^* > a^*$. Thus, one can meet a case of cyclic-plastic collapse with existing elastic fields where plastic strains velocities $\dot{\boldsymbol{\Theta}}_p = \boldsymbol{0}$. The theorem of the cyclic-plastic collapse, on the basis of which is constructed the mathematical model (34)–(36), does not require a satisfying of the criterion (10).

The plastic multipliers velocities can be obtained directly by applying the Rosen project gradient method for the problem (34)–(36) analysis. The type of collapse is identified having performed analysis of the solution [19]. The mathematical model (34)–(36) subsequently will be incorporated into structural unit of the iterative algorithm, developed for approximate analysis of adapted bending plate optimization problem.

4. The problem of plate parameters distribution at shakedown

4.1. Mathematical model of the problem

The adapted plate satisfies strength (yield) conditions and is safe in respect to cyclic-plastic collapse [22]. However, residual displacements $\boldsymbol{u}_r$ can exist in the plate with developed plastic strains, even if loading is equal to zero. Sometimes residual displacements can be significantly large even causing exploitation unsuitability of the structure (indeed in most cases the total deflections $\boldsymbol{u} = \boldsymbol{u}_e + \boldsymbol{u}_r$ should be verified). Therefore it is important to define not only the stress state but also the strain state of the flexural plates at shakedown. The main mathematical models, constructed for optimization problems with strength and stiffness constraints at shakedown, are presented in the paper [23]. On the basis of these models the following mathematical model for determining optimal distribution of the parameters of adapted plate is constructed
find

$$min\ \sum_k \boldsymbol{L}_k \boldsymbol{M}_{0k} = \ min\ \ \boldsymbol{L}^T \boldsymbol{M}_0 \quad (37)$$

subject to

$$min\ \frac{1}{2}\sum_k \boldsymbol{M}_{rk}^T \boldsymbol{D}_k \boldsymbol{M}_{rk} = min\ \frac{1}{2}\boldsymbol{M}_r^T \boldsymbol{D}\, \boldsymbol{M}_r \quad (38)$$

$$\boldsymbol{A}\boldsymbol{M}_r = \boldsymbol{0} \quad (39)$$

$$\varphi_{kl,j} = (M_{0k})^2 - (\boldsymbol{M}_{ekl,j} + \boldsymbol{M}_{rkl})^T \boldsymbol{\Pi}_{kl} (\boldsymbol{M}_{ekl,j} + \boldsymbol{M}_{rkl}) \geq 0 \quad (40)$$

$$\boldsymbol{M}_{0k} \geq \boldsymbol{0} \quad (41)$$

$$\boldsymbol{u}_{r,\,min} \le \boldsymbol{H}\boldsymbol{\Theta}_p \le \boldsymbol{u}_{r,\,max} \tag{42}$$

$$\boldsymbol{\Theta}_p = \left(\boldsymbol{\Theta}_{pkl}\right)^T,\;\; \boldsymbol{\Theta}_{pkl} = \sum_j \left[\nabla\boldsymbol{\varphi}_j\left(\boldsymbol{M}_{ej} + \boldsymbol{M}_r\right)\right]^T \boldsymbol{\lambda}_j \tag{43}$$

$$\boldsymbol{\lambda}_j = \left(\lambda_{kl,j}\right)^T,\;\; \lambda_{kl,j} \ge 0 \tag{44}$$

Let us consider the contents of the mathematical model (37)–(44) assigned for the plate optimization problem. The components of the vector $\boldsymbol{L} = (L_1, L_2, ..., L_s)^T$ are the areas of finite elements of the plate discrete model. Though objective function (37) matches with the expression (34), there is no meaning of physical energy dissipation rate in the problem (37)–(44). This meaning "returns" if the stiffness conditions (42)–(44) are ignored in the mathematical model (37)–(44). Then the cyclic-plastic collapse conditions are obtained for optimization problem (34)–(36) (criterion (38) ensures only statically admissible residual bending moments $\boldsymbol{M}_r^*$ at cyclic-plastic collapse time and also minimizes the value of complimentary deformation energy).

The main unknowns on the problem (37)–(44) are vectors of limit $\boldsymbol{M}_0$ and residual $\boldsymbol{M}_r$ bending moments and the vector plastic multipliers $\boldsymbol{\lambda}_j$ ($j \in J$). The vectors $\boldsymbol{u}_{r,\,min}$, $\boldsymbol{u}_{r,\,max}$ are known in advance and describe the variation of the residual displacements $\boldsymbol{u}_r$. As it was mentioned above, in case of constraining the total displacements the stiffness conditions (42) take the following form:

$$\boldsymbol{u}_{min} \le \boldsymbol{u}_{e,\,inf} + \boldsymbol{u}_r\,,\quad \boldsymbol{u}_{e,\,sup} + \boldsymbol{u}_r \le \boldsymbol{u}_{max} \tag{45}$$

here the plate residual displacements are defined via $\boldsymbol{u}_r = \boldsymbol{H}\boldsymbol{\Theta}_p$.

However, the mathematical model (37)–(44) strictly speaking is not exhaustive (entirely completed).

1. Thought when solving auxiliary problem (38)–(40) plastic multipliers $\boldsymbol{\lambda}$ can be obtained (Eq. (20)): $\boldsymbol{\lambda} = \left(\nabla\boldsymbol{\varphi}\nabla^T\boldsymbol{\varphi}\right)^{-1}\nabla\boldsymbol{\varphi}\nabla\mathcal{F}$), the fact that these multipliers are obtained is not well-defined. The relation between residual bending moments $\boldsymbol{M}_r$ and plastic strain $\boldsymbol{\Theta}_p$ (i.e. plastic multipliers λ) was not employed in the mathematical model (37)–(44) as it is given in the formula (22).
2. Residual displacements (deflections) $\boldsymbol{u}_r$ at shakedown vary nonmonotonously. In other words, shakedown state (42) can be reached when the distribution of residual displacements is not unique. It is especially relevant for the beam structures and partially relevant for the plates too.

Thus, there can be several variants of the mathematical model to determine optimal distribution of plate parameters. The decision which should be applied depends on mathematical programming experience of the researcher.

4.2. Determination of variation bounds of residual displacements

Residual internal forces $\boldsymbol{M}_r$ emerge under the influence of elastic-plastic strains in adapted structure. These internal forces ensure that new plastic strains $\boldsymbol{\Theta}_p$ will not develop from load variation. In general case the distribution of internal forces $\boldsymbol{M}_r$ of the adapted structure is not unique: it depends on the particular loading history. The residual displacements $\boldsymbol{u}_r$ depend on this history too. For the plastic strains, emerging moment at the j-th design section, the following dependency is valid

$$\varphi_{kl} = 0\,,\quad \lambda_{kl,j}\,\varphi_{kl,j} = 0\,,\quad \lambda_{kl,j} > 0 \tag{46}$$

The value of plastic multiplier $\lambda_{kl,j} > 0$ varies during subsequent deformation process when slackness conditions (46) are satisfied, but remains the non-zero value till the end of loading process. During the plastic deformation process an unloading phenomenon of the cross-section is possible: at some deformation stages yield condition is satisfied as equality, i.e. $\varphi_{kl,j} = 0$ for j - th cross-section, in subsequent deformation stages it changes to inequality $\varphi_{kl,j} > 0$. So slackness conditions (46) are violated $\varphi_{kl} = 0$, $\lambda_{kl,j}\,\varphi_{kl,j} = 0$, $\lambda_{kl,j} > 0$. At the solitary instance when the state of the structure is near cyclic-plastic collapse the distribution of residual internal forces $\boldsymbol{M}_r^*$, obtained by solving the problem (10)–(13), is unique for each of the loading histories $\boldsymbol{F}_{inf} \le \boldsymbol{F}(t) \le \boldsymbol{F}_{sup}$. However the distribution of residual displacements $\boldsymbol{u}_r$ still can be nonunique. Such a proposition is predetermined by the above-mentioned unloading phenomenon of the sections and the variation of nonmonotonous residual displacements during loading process [19]. Minimum and maximum values of displacement vectors $\boldsymbol{u}_{r,\,inf}$, $\boldsymbol{u}_{r,\,sup}$, these being not related to the time t, are introduced for the evaluation of nonmonotonous variation of the residual displacements. The displacements bounds vectors $\overline{\boldsymbol{u}}_{r,inf}^*$, $\overline{\boldsymbol{u}}_{r,sup}^*$ are obtained by analyzing the all possible loading histories $\boldsymbol{F}(t)$. Meanwhile the vectors $\boldsymbol{u}_{r,\,inf}$, $\boldsymbol{u}_{r,\,sup}$ are rather approximate comparing with safe bounds of residual displacement, defined by

$$\boldsymbol{u}_{r,\,inf} \le \overline{\boldsymbol{u}}_{r,\,inf}^*\,,\quad \overline{\boldsymbol{u}}_{r,\,sup}^* \le \boldsymbol{u}_{r,\,sup} \tag{47}$$

Further the mathematical model of bounds determination of residual displacements variation is formulated as the mathematical programming problem. The objective function of the problem depends on plastic strains, the constraints of this problem represent the static and kinematical admissibility conditions of residual displacements and strains.

The first problem. The **c**omponents $\widetilde{u}_{ri,\,inf}$, $\widetilde{u}_{ri,\,sup}$ ($i = 1, 2, ..., m$) of the vector of kinematical residual displacements $\boldsymbol{u}_r$ are obtained via the solution of the following linear mathematical programming problem

find

$$\begin{matrix} max \\ min \end{matrix}\; \widetilde{\boldsymbol{H}}_i\;\widetilde{\lambda} = \begin{bmatrix} \widetilde{u}_{ri,\,sup} \\ \widetilde{u}_{ri,\,inf} \end{bmatrix},\quad i = 1, 2, ..., m \tag{48}$$

subject to

$$\boldsymbol{B}_{\lambda}\,\widetilde{\boldsymbol{\lambda}} = \boldsymbol{B}_r \boldsymbol{M}_r^*, \quad \widetilde{\boldsymbol{\lambda}} \geq \boldsymbol{0} \tag{49}$$

$$\widetilde{\boldsymbol{\lambda}}^T \widetilde{\boldsymbol{C}} \leq \widetilde{D}_{max} \tag{50}$$

This mathematical model corresponds to fictitious structure, i.e. the displacements $\widetilde{\boldsymbol{u}}_{r,inf}$, $\widetilde{\boldsymbol{u}}_{r,sup}$ at shakedown state "envelope" the displacements $\boldsymbol{u}_r$ of the given structure [19]. The distribution of residual internal forces $\boldsymbol{M}_r^*$ for this structure is unique for any of loading histories $\boldsymbol{F}_{inf} \leq \boldsymbol{F}(t) \leq \boldsymbol{F}_{sup}$. Unknowns of the problem (48)–(50) are the components of ζ - size vector $\widetilde{\boldsymbol{\lambda}}$, keeping in mind that the vectors $\boldsymbol{M}_r^*$, $\widetilde{\boldsymbol{C}}$ and the value D_{max} are known. Further the solution algorithm of mathematical model (48)–(50) will be discussed in more details.

Thus, the vector $\boldsymbol{M}_r^*$ of the initial system defined by the limit bending moment vector $\boldsymbol{C}$ is obtained for the known RVL bounds $\boldsymbol{F}_{inf}$, $\boldsymbol{F}_{sup}$. Further, having introduced the new plasticity constants vector $\widetilde{\boldsymbol{C}}$, a fictitious system is constructed. The vector $\widetilde{\boldsymbol{C}}$ shows that one yield condition for the i-th plate design section is active, i.e. at least one condition is satisfied as a strict equality: $\varphi_{kl,j} = \widetilde{C}_k - f_{kl}\left(M_{ekl,j} + M_{rkl}\right) = 0$. The limit bending moment $\widetilde{C}_k$, corresponding to design section of the fictitious plate, is calculated by the following formula

$$\widetilde{C}_k = max\ \boldsymbol{f}\left(\boldsymbol{M}_{ej} + \boldsymbol{M}_r^*\right) \geq 0, \quad k \in K, \quad j \in J \tag{51}$$

The vector of elastic internal forces $\boldsymbol{M}_{ej}^*$ of the yield conditions $\varphi_{kl} = C_k - \boldsymbol{M}_{kl}^T \boldsymbol{\Pi}_{kl}\, \boldsymbol{M}_{kl} \geq 0$, which are satisfied by equality (51), and the vector $\widetilde{\boldsymbol{C}}$ are defined simultaneously. Then the following equality is valid

$$\widetilde{C}_k = \boldsymbol{f}\left(\boldsymbol{M}_{ej}^* + \boldsymbol{M}_r^*\right) \tag{52}$$

It means that unloading phenomenon of the fictitious elastic-plastic system sections will not occur for any loading history $\boldsymbol{F}(t)$ within load variation bounds $\boldsymbol{F}_{inf} \leq \boldsymbol{F}(t) \leq \boldsymbol{F}_{sup}$.

The upper bound of dissipated energy $\widetilde{D}_{max}$ during the shakedown process is obtained according to the optimal solution of the problem (14)–(16). The dissipated energy D_{max} also can be calculated applying the formula suggested by Koiter [24]. However, the method of fictitious structure allows evaluating the residual displacements variation bounds $\widetilde{\boldsymbol{u}}_{r,inf}$, $\widetilde{\boldsymbol{u}}_{r,sup}$ more exactly comparing with the ones obtained via Koiter's global conditions.

The matrix $\widetilde{\boldsymbol{H}}$ employed in the objective function (48) is calculated according the formula $\tilde{\boldsymbol{H}} = \boldsymbol{H}\left[\nabla\boldsymbol{\varphi}\left(\boldsymbol{M}_{ek}^* + \boldsymbol{M}_{rk}^*\right)\right]$. The equalities (49) $\boldsymbol{B}_{\lambda}\widetilde{\boldsymbol{\lambda}} = \boldsymbol{B}_r\boldsymbol{M}_r^*$ correspond to the plate compatibility equations

$$\boldsymbol{B}\boldsymbol{\Theta}_p = \boldsymbol{B}_r \boldsymbol{M}_r^* \tag{53}$$

They are obtained by eliminating the residual displacements $\boldsymbol{u}_r$ from the geometrical equations (15). Aiming to create the matrices $\boldsymbol{B}$ and $\boldsymbol{B}_r$ the matrix $\boldsymbol{A}^T$ is divided into two sub-matrices, namely: quadratic matrix $\boldsymbol{A}'^T$ (for which exist inverse matrix) and the rest part, denoted via $\boldsymbol{A}''^T$. The same operation (decomposition into two parts) is performed for the flexibility matrix $\boldsymbol{D}$ and the vector of plastic strains $\boldsymbol{\Theta}_p$. Compatibility equations of geometrical strains and residual displacements then read

$$\boldsymbol{A}'^T \boldsymbol{u}_r = \boldsymbol{D}'^T \boldsymbol{M}_r + \boldsymbol{\Theta}'_p$$

$$\boldsymbol{A}''^T \boldsymbol{u}_r = \boldsymbol{D}''^T \boldsymbol{M}_r + \boldsymbol{\Theta}''_p$$

The expression $\boldsymbol{u}_r = \left(\boldsymbol{A}'^T\right)^{-1}\left(\boldsymbol{D}'\boldsymbol{M}_r + \boldsymbol{\Theta}'_p\right)$, being derived from the first equality, and the unit matrix $\boldsymbol{I}$ are introduced into the second equality. So, the equality (53) is obtained, where the matrices $\boldsymbol{B}$ and $\boldsymbol{B}_r$ are expressed by

$$\boldsymbol{B} = \left(\boldsymbol{A}''^T\left(\boldsymbol{A}'^T\right)^{-1}, -\boldsymbol{I}\right)$$

$$\boldsymbol{B}_r = -\boldsymbol{A}''^T\left(\boldsymbol{A}'^T\right)^{-1}\boldsymbol{D}' + \boldsymbol{D}''$$

The optimal solution of problem the (48)–(50) is vector $\widetilde{\boldsymbol{\lambda}}^* \geq \boldsymbol{0}$ components. This is another approach different to the one of the problem (14)–(16), may not represent the physical meaning of plastic multipliers.

The second problem. The principle of complementary energy minimum and the compatibility equations (53) for strains of elastic-plastic system are adequate. Thus, the problem of residual displacements variation bounds can be analyzed by applying the basic solution vectors $\boldsymbol{\lambda}_0 \geq \boldsymbol{0}$ of the strain compatibility equations

$$\boldsymbol{B}_{0\lambda}\boldsymbol{\lambda}_0 = \boldsymbol{B}_r \boldsymbol{M}_r^* \tag{54}$$

Basic variables $\boldsymbol{\lambda}'_0 \geq \boldsymbol{0}$ of the vector $\boldsymbol{\lambda}_0 \geq \boldsymbol{0}$ can be determined according to the formula $\lambda'_0 = \left(\boldsymbol{B}'^*_{\lambda}\right)^{-1}\boldsymbol{B}_r\boldsymbol{M}_r^*$. Here quadratic $\left(k_0 \times k_0\right)$ matrix $\boldsymbol{B}'^*_{\lambda}$ is the sub-matrix one of $\boldsymbol{B}^*_{\lambda}$. If determinant of the matrix $\boldsymbol{B}'^*_{\lambda}$ is equal to zero, the statically determinate system corresponding to $\boldsymbol{B}'^*_{\lambda}$ is geometrically unstable. Generally, the number η of the combinations, those constructing the sub-matrices $\boldsymbol{B}'^*_{\lambda}$, can be smaller or equal to $\zeta!/\left[k_0!\left(\zeta - k_0\right)!\right]$. After all ω_{λ} vectors $\boldsymbol{\lambda}_0 \geq \boldsymbol{0}$ (here subscript ω_{λ} is omitted for vector $\boldsymbol{\lambda}_0$) are found, selected are only satisfying energy condition (49) vectors. Denote the set of the vectors $\boldsymbol{\lambda}_{0,\omega} \geq \boldsymbol{0}$ subscripts $\omega = 1, 2, ..., \omega_{\lambda}$ via Ω. The residual displacements vectors $\boldsymbol{u}_{r0,\omega}$ are calculated according to the following formula

$$\boldsymbol{u}_{r0,\,\omega} = \boldsymbol{H}^{*}\boldsymbol{\lambda}_{0,\,\omega}\,, \quad \omega \in \Omega \tag{55}$$

The vectors $\boldsymbol{u}_{r,\,inf}$, $\boldsymbol{u}_{r,\,sup}$ are constructed by picking components of all vectors $\boldsymbol{u}_{r0,\,\omega}$ ($\omega \in \Omega$) with maximum and minimum values. It is easy to find that one of the vectors $\boldsymbol{\lambda}_{0,\,\omega} \geq \boldsymbol{0}$ will coincide with optimal solution $\boldsymbol{\lambda}^{*} \geq \boldsymbol{0}$ of the problem (14)–(16). Thus, it is possible to write a group of inequalities

$$\widetilde{\boldsymbol{u}}_{r,\,inf} \leq \boldsymbol{u}_{r,\,inf} \leq \boldsymbol{u}_{r}(t) \leq \boldsymbol{u}_{r,\,sup} \leq \widetilde{\boldsymbol{u}}_{r,\,sup} \tag{56}$$

The following sequence of inequalities is obtained taking in to account the inequalities (47)

$$\widetilde{\boldsymbol{u}}_{r,\,inf} \leq \boldsymbol{u}_{r,inf} \leq \overline{\boldsymbol{u}}^{*}_{r,\,inf} \leq \boldsymbol{u}_{r}(t) \leq \overline{\boldsymbol{u}}^{*}_{r,\,inf} \leq \boldsymbol{u}_{r,\,sup} \leq \widetilde{\boldsymbol{u}}_{r,\,sup} \tag{57}$$

The residual strains compatibility equations (49)

$$\boldsymbol{B}_{\lambda}\widetilde{\boldsymbol{\lambda}} = \boldsymbol{B}_{r}\boldsymbol{M}^{*}_{r}\,, \quad \widetilde{\boldsymbol{\lambda}} \geq \boldsymbol{0}$$

which are included in constraints of the residual displacements variation bounds of the optimization problem (48)–(50), can be obtained applying the formulae $\boldsymbol{G}\boldsymbol{\Theta}_{p} = \boldsymbol{M}_{r}$, $\boldsymbol{\Theta}_{pkl} = \sum_{j}\left[\nabla\boldsymbol{\varphi}_{j}\left(\boldsymbol{M}_{ej} + \boldsymbol{M}_{r}\right)\right]^{T}\widetilde{\boldsymbol{\lambda}}_{j}$ and the matrix $\boldsymbol{B}_{r} = -\boldsymbol{A}''^{T}\left(\boldsymbol{A}'^{T}\right)^{-1}\boldsymbol{D}' + \boldsymbol{D}''$. Then

$$\boldsymbol{G}\left[\nabla\boldsymbol{\varphi}\left(\boldsymbol{M}^{*}\right)\right]^{T}\widetilde{\boldsymbol{\lambda}} = \boldsymbol{M}^{*}_{r} \tag{58}$$

$$\boldsymbol{B}_{r}\,\boldsymbol{G}\left[\nabla\boldsymbol{\varphi}\left(\boldsymbol{M}^{*}\right)\right]^{T}\widetilde{\boldsymbol{\lambda}} = \boldsymbol{B}_{r}\boldsymbol{M}^{*}_{r} \tag{59}$$

Then strain compatibility equation is obtained by

$$\boldsymbol{B}_{\lambda}\widetilde{\boldsymbol{\lambda}} = \boldsymbol{B}_{r}\boldsymbol{M}^{*}_{r}$$

here matrix $\boldsymbol{B}^{*}_{\lambda} = \boldsymbol{B}_{r}\,\boldsymbol{G}\left[\nabla\boldsymbol{\varphi}\left(\boldsymbol{M}^{*}\right)\right]^{T}$.

It is possible to change the constraints (49) of the residual displacements variation bounds optimization problem (48)–(50) by the condition (58)

$$\boldsymbol{G}\left[\nabla\boldsymbol{\varphi}\left(\boldsymbol{M}^{*}\right)\right]^{T}\widetilde{\boldsymbol{\lambda}} = \boldsymbol{M}^{*}_{r}\,, \quad \widetilde{\boldsymbol{\lambda}} \geq \boldsymbol{0} \tag{60}$$

having eliminated the linearly dependent equations in advance. However, it is more practical to use the compatibility equations of the residual strains (49): physical meaning of the second problem of residual displacements variation bounds $\boldsymbol{u}_{r,\,inf}$, $\boldsymbol{u}_{r,\,sup}$ determination becomes then evident. Both the vectors $\boldsymbol{u}_{r,\,inf}$, $\boldsymbol{u}_{r,\,sup}$ and $\widetilde{\boldsymbol{u}}_{r,inf}$, $\widetilde{\boldsymbol{u}}_{r,\,sup}$ can be incorporated into stiffness constraints (57) of mathematical models of the optimization problem.

4.3. The modified model of optimization problem

The model is similar to that of (37)–(44), only the member $u_{ri,inf} = min\,\widetilde{\boldsymbol{H}}_{i}\widetilde{\boldsymbol{\lambda}}$, $u_{ri,sup} = max\,\widetilde{\boldsymbol{H}}_{i}\widetilde{\boldsymbol{\lambda}}$ is introduced into the condition (42)

find

$$min\sum_{k} L_{k}M_{0k} = min\,\boldsymbol{L}^{T}\boldsymbol{M}_{0} \tag{61}$$

subject to

$$min\,\frac{1}{2}\sum_{k}\boldsymbol{M}^{T}_{rk}\boldsymbol{D}_{k}\boldsymbol{M}_{rk} = min\,\frac{1}{2}\boldsymbol{M}^{T}_{r}\boldsymbol{D}\,\boldsymbol{M}_{r} \tag{62}$$

$$\boldsymbol{A}\boldsymbol{M}_{r} = \boldsymbol{0} \tag{63}$$

$$\varphi_{kl,j} = \left(M_{0k}\right)^{2} - \left(\boldsymbol{M}_{ekl,j} + \boldsymbol{M}_{rkl}\right)^{T}\boldsymbol{\Pi}_{kl}\left(\boldsymbol{M}_{ekl,j} + \boldsymbol{M}_{rkl}\right) \geq 0 \tag{64}$$

$$M_{0k} \geq 0 \tag{65}$$

$$\boldsymbol{u}_{r,min} \leq \boldsymbol{H}\boldsymbol{\Theta}_{p} \leq \boldsymbol{u}_{r,max} \tag{66}$$

$$\boldsymbol{\Theta}_{p} = \left(\boldsymbol{\Theta}_{pkl}\right)^{T}, \quad \boldsymbol{\Theta}_{pkl} = \sum_{j}\left[\nabla\boldsymbol{\varphi}_{j}\left(\boldsymbol{M}_{ej} + \boldsymbol{M}_{r}\right)\right]^{T}\boldsymbol{\lambda}_{j} \tag{67}$$

$$\boldsymbol{\lambda}_{j} = \left(\lambda_{kl,j}\right)^{T};\ \lambda_{kl,j} \geq 0\,,\ k \in K\,,\ l \in L\,,\ j \in J \tag{68}$$

$$u_{ri,\,inf} = \min_{v}\,\widetilde{\boldsymbol{H}}_{i}\,\widetilde{\boldsymbol{\lambda}}\,, \quad u_{ri,\,sup} = \max_{v}\,\widetilde{\boldsymbol{H}}_{i}\widetilde{\boldsymbol{\lambda}}$$
$$i = 1,\,2,\,...,\,m \tag{69}$$

$$\boldsymbol{B}_{\lambda}\widetilde{\boldsymbol{\lambda}} = \boldsymbol{B}_{r}\boldsymbol{M}^{*}_{r}\,, \quad \sum_{j}\widetilde{\boldsymbol{\lambda}}^{T}_{j}\,\widetilde{\boldsymbol{C}} \leq \widetilde{D}_{max}\,, \quad \widetilde{\boldsymbol{\lambda}} \geq \boldsymbol{0} \tag{70}$$

$$\boldsymbol{u}_{r,\,min} \leq \boldsymbol{u}_{r,\,inf}\,, \quad \boldsymbol{u}_{r,\,sup} \leq \boldsymbol{u}_{r,\,max} \tag{71}$$

The shortage of this model is that it is the verification one: in fact the cyclic-plastic collapse problem (34)–(36) is analyzed, but as distinct from the optimal bending plate project, there are verified the stiffness conditions (42) or (69)–(71) of certain accuracy at cyclic-plastic collapse (in each Rosen algorithm step). In other words, the main solution of the optimization problem is not directly influenced by stiffness conditions.

One must note, that when applying the nonlinear Mizes yield condition, the residual bending moments have influence to the matrix $\boldsymbol{G}$ $\boldsymbol{G} = \boldsymbol{K}\boldsymbol{A}^{T}\left(\boldsymbol{A}\boldsymbol{K}\boldsymbol{A}^{T}\right)^{-1}\boldsymbol{A}\boldsymbol{K} - \boldsymbol{K}$ depending on $\boldsymbol{M}_{r}$ ($\boldsymbol{M}_{r} = \boldsymbol{G}\boldsymbol{\Theta}_{p}$, $\boldsymbol{\Theta}_{p} = \left(\boldsymbol{\Theta}_{pkl}\right)^{T}$, $\boldsymbol{\Theta}_{pkl} = 2\sum_{j}\lambda_{kl,j}\boldsymbol{\Pi}_{kl}\boldsymbol{M}_{kl.j} = \left[\nabla\varphi_{kl,j}\left(\boldsymbol{M}_{kl,j}\right)\right]^{T}\lambda_{kl,j}$). Here $\boldsymbol{A}$ and $\boldsymbol{K}$ are the equilibrium and physical matrices, respectively. Thus, the residual bending moments and displacements matrices $\boldsymbol{G}$ and $\boldsymbol{H}$ are determined standing behind of one iteration of analysis problem.

The first modified model of optimization problem is solved by stages (the solution algorithm scheme is presented in Fig. 1) freely introducing the objective function (37) variation step.

4.4. Numerical example of iterative solving algorithm of the plate optimization problem

Let discuss the features of the algorithm of the analysis problem (61)–(71). One can find that the problem (61)–(71) is not a classical mathematical programming problem. Its composition includes the separate quadratic programming problem (62)–(64) corresponding to analysis problem (10)–(13) for the residual bending moments $\boldsymbol{M}_r$. The distribution of optimal parameters of plate at cyclic-plastic collapse will be obtained differently (without an employing he constrains (66)–(68) and (69)–(71)).

Obviously the solution of the problem (61)–(71) is not reached per one iteration. Thus, one of the possible solution ways of plate optimization problem under presence of stiffness constraints is application of an iterative solving.

An application of iterative algorithm the problem (61)–(71) solving is presented via numerical solution of plate (see Fig. 1).

The circular plate of radius $R = 0.90m$, supported via hinges per outer contour is under considered (see Fig. 2). The material of the plate is steel (material physical properties are presented in Fig. 2). Plate is subjected by symmetrically and uniformly distributed loading q and uniformly distributed bending moment M, applied at the plate outer contour.

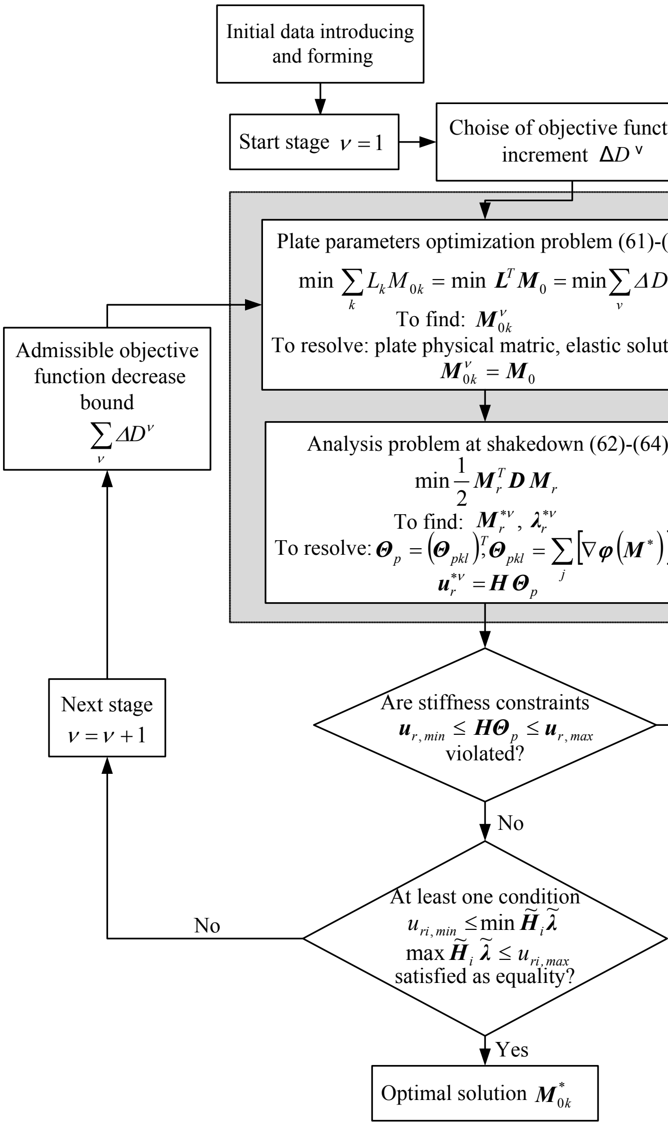
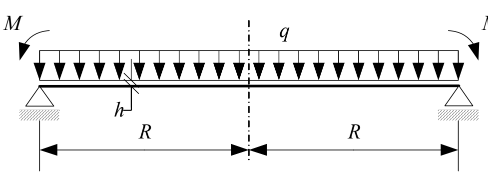


Fig. 1 Iterative solving algorithm of the plate optimization problem



σ_y	210	[MPa]
E	210	[GPa]
ν	1/3	-
h	0.03	[m]
R	0.9	[m]
q_{inf}	100	[kN/m^2]
q_{sup}	95	[kN/m^2]
M_{inf}	36.25	[kN]

Fig. 2 Plate load diagram and initial data

The initial plate cross-section height is $h = 0.03\,\mathrm{m}$. Elastic bending moments $\boldsymbol{M}_{e,\,min}$, $\boldsymbol{M}_{e,\,max}$ of the plate initial cross-section are calculated applying the internal forces matrix $\boldsymbol{\alpha}$ of elastic analysis. Then $\boldsymbol{M}_{e,min} = \boldsymbol{\alpha}_{min}\boldsymbol{q}_{min}$, $\boldsymbol{M}_{e,max} = \boldsymbol{\alpha}_{max}\boldsymbol{q}_{max}$. When the total internal forces at the yield conditions (64) are calculated the elastic bending moments $\boldsymbol{M}_e$, being distributed at the contour, are treated subsequently as constants.

Optimal solution of the problem is presented in the last row of the Table.

Table

Convergence of the limit bending moments (kN) of plate elements

STAGE	$M_{0,1}$	$M_{0,2}$	$M_{0,3}$	$M_{0,4}$	$M_{0,5}$	$M_{0,6}$	$\boldsymbol{L}^T\boldsymbol{M}_0$
$v = 1$	53.06	52.77	52.53	52.29	52.05	51.81	132.78
$v = 2$	52.92	52.18	51.40	49.80	48.85	47.90	125.93
$v = 3$	52.91	52.10	51.35	49.74	48.43	47.39	125.21
$v = 4$	52.87	51.91	51.23	49.61	47.39	46.11	123.40
$v = 5$	52.77	51.71	51.07	49.39	47.08	44.23	121.52
$v = 6$	52.37	51.56	50.65	48.72	46.02	42.64	119.07

5. Conclusions

The mathematical model of flexural plate optimization problem at shakedown conjoins the verification problem for plate analysis and the verification of stiffness conditions. Algorithms for the solution of two problems are presented in current investigation. The more exact results are obtained when plastic multipliers as the main unknowns are introduced. It is evident that the suggested and subsequently developed algorithms for solving the investigated class of problems should be introduced into the software for the analysis of object-oriented structures.

References

1. **Weichert, D., Maier G., editors.** Inelastic Analysis of Structures under Variable Loads. Theory and Engineering Applications.-Dordrecht: Kluwer Academic Publishers, 2002.-396p.
2. **Capurso, M.** A Displacement bounding principle in shakedown of structures subjected to cyclic loads.-Int. J. of Solids and Structures, 1974, v.10, p.77-92.
3. **Dorosz, St.** An improved bound to maximum deflections of elastic-plastic structures at shakedown.-J. of Structural Mechanics, 1978, v.6, p.267-287.
4. **Čyras, A.** Mathematical Models for the Analysis and Optimization of Elastoplastic Structures.-New York: John Wiley&Sons Ltd., 1983.-121p.
5. **Kleiber, M., Konig, J. A., editors.** Inelastic Solid and Sructures. Antoni Sawczuk Memorial Volume.-Swansea: Pineridge Press, 1990.-558p.
6. **Stein, E., Zhang, G., Mahnken, R.** Shakedown analysis for perfectly plastic and kinematic hardening materials.-In: Stein, E., editor. Progress in Computational Analysis in Inelastic Structures.-Wien: Springer-Verlag, 1993, p.175-244.
7. **Giambanco, F., Palizzolo, L., Polizzotto, C.** Optimal shakedown design of circular plates.-ASCE J. of the Engineering Mechanics Division, 1994, v.120, p.2535-2355.
8. **Kaliszky, S., Logo, J.** Optimal plastic limit and shakedown design of bar structures with constraints on plastic deformation.-Engineering Structures, 1997, v.19, p.19-27.
9. **Tin-Loi, F.** Optimum shakedown design under residual displacement constraints.-Structural and Multidisciplinary Optimization, 2000, v.19, p.130-139.
10. **Aliavdin, P. V.** Limit Analysis of Structures under Variable Loads.- Minsk: UP Technoprint, 2005.-282p. (in Russian).
11. **Atkočiūnas, J., Jarmolajeva, E., Merkevičiūtė, D.** Optimal shakedown loading for circular plates.-Structural and Multidisciplinary Optimization, 2004, v.27, No3, p.178-188.
12. **Atkočiūnas, J., Rimkus, L., Jarmolajeva, E.** Shakedown loading optimization for circular plates.-Metal Structures–Design, Fabrication, Economy. Proceedings of the International Conference on Metal Structures-ICMS-03. Miscolc, Hungary, April 3-5, Rotterdam: Millpress, 2003, p.251-258.
13. **Bazaraa, M. S., Sherali, H. D., Shetty, C. M.** Nonlinear Programming: Theory and Algorithms.-New York: John Wiley & Sons Ltd., 2004.-853p.
14. **Chraptovič, E., Atkočiūnas, J.** Mathematical programming applications peculiarities in shakedown problem.-Statyba, 2001, No2, p.106-114. (in Russian).
15. **Gallagher, R. H.** Finite Element Analysis: Fundamentals.-Englewood Clifts: Prentice-Hall Inc., 1975.-420p.
16. **Kalanta, S.** Elastic plates and shells. In: Computeraided Calculation of Structural Mechanics Problems.-Vilnius: Science and Encyclopaedic Publishing House, 1994, p.163-218. (in Lithuanian).
17. **Jarmolajeva, E., Atkočiūnas, J.** Shakedown loading optimization under constrained residual displacements–formulation and solution for circular plates.-J. of Civil Engineering and Management, 2002, v.VIII, No1, p.54-67.
18. **Maier, G.** A matrix structural theory of piecewise-linear plasticity with interacting yield planes.-Meccanica, 1970, No7, p.51-66.
19. **Atkočiūnas, J.** Design of Elastoplastic Systems under Repeated Loading.-Vilnius: Science and Encyclopedic Publishing House, 1994.-148p. (in Russian).
20. **Prager, W., Shield, R. T.** A general theory of optimal plastic design. ASME Transactions Series E Journal of Applied Mechanics, 1967, v.34, p.184-186.
21. **Borkowski, A., Atkočiūnas, J.** Optimal design for cyclic loading.-In: Sawczuk A., Mróz Z., editors. Op-

timization in Structural Design.-Berlin: Springer-Verlag, 1975, p.433-440.
22. **Pham, D. C.** Plastic collapse of a circular plate under cyclic loads.-International Journal of Plasticity, 2003, v.19, p.547-559.
23. **Atkočiūnas, J.** Mathematical models of optimization problems at shakedown.-Mechanics Research Communications, 1999, v.26, p.319-326.
24. **Koiter, W.** General theorems for elastic-plastic solids.-Progress in Solid Mechanics. Amsterdam: North-Holland, 1960, p.165-221.

J. Atkočiūnas, L. Rimkus, V. Skaržauskas, E. Jarmolajeva

PLOKŠČIŲ OPTIMIZAVIMAS PRISITAIKOMUMO SĄLYGOMIS

R e z i u m ė

Straipsnyje nagrinėjamos tampriosios plastinės lenkiamos metalinės plokštės, veikiamos kintamos kartotinės apkrovos. Plokštės geometrijos forma yra žinoma, apkrova apibūdinama tik viršutinėmis ir apatinėmis nuo laiko nepriklausančiomis kitimo ribomis. Ieškoma optimalaus plokštės projekto, atsižvelgiant į plokštės stiprumo ir standumo reikalavimus. Optimizuojamais parametrais laikomi ribinis plokštės momentas ar būdingas skerspjūvio matmuo. Siūlomas naujas iteracinis prisitaikančių lenkiamų plokščių optimizavimo uždavinių apytikrio sprendimo algoritmas, kurio pagrindas yra Rozeno projektuojamųjų gradientų metodas. Tuo tikslu sudarytas lenkiamos plokštės prisitaikomumo būvio įtempių ir deformacijų skaičiavimo (analizės) uždavinio statinės formuluotės matematinis modelis. Siūlomas algoritmas iliustruojamas apvalios plokštės optimalaus projekto uždavinio sprendimu. Tyrimai atlikti ir skaitinių eksperimentų rezultatai gauti, laikantis mažų poslinkių prielaidos.

J. Atkočiūnas, L. Rimkus, V. Skaržauskas, E. Jarmolajeva

OPTIMAL SHAKEDOWN DESIGN OF PLATES

S u m m a r y

In this paper the optimal shakedown of perfectly elastic-plastic bending metallic plates with strength and stiffness constraints are considered. The geometry of the plate and its acting variable repeated load are known. Here optimal distribution of limit bending moments or characteristic dimension of cross-section for adapted bending plate is to be found. In the paper a new iterative approximate solution algorithm based on Rosen project gradient is proposed for optimal shakedown design of the plates. While solving the static formulation of the analysis problem their dual (kinematic formulation) solution is determined by using the Rosen criterion mathematical-mechanical interpretation, which is explained before by the authors. The solution algorithm is illustrated by the numerical example of optimal project calculation of circular plate. The investigations are performed and results of numerical experiments are obtained according to assumptions of small displacements.

Ю. Аткочюнас, Л. Римкус, В. Скаржаускас, Э. Ярмолаева

ОПТИМИЗАЦИЯ ПЛАСТИН В УСЛОВИЯХ ПРИСПОСОБЛЯЕМОСТИ

Р е з ю м е

Рассматриваются идеально упруго-пластические металлические пластины заданной геометрии под действием повторно-переменной нагрузки. Нагрузка характеризуется только независящими от времени верхними и нижними пределами ее изменения. Задача оптимизации параметров пластины реализуется при наличии как прочностных, так и жесткостных ограничений. Оптимизируемые параметры – предельные усилия изгибаемых пластин либо характерный размер сечения. Предлагается новый итерационный алгоритм решения проектной задачи оптимизации пластин в условиях приспособляемости с использованием алгоритма проектируемых градиентов Розена. С этой целью строится математическая модель задачи определения напряженно-деформируемого состояния изгибаемых пластин (задача анализа) в статической формулировке. Предлагаемый алгоритм иллюстрируется примером решения задачи оптимизации круглой пластины. Пластина рассчитывается в соответствии с упрощенной технической теорией вместе с предпосылкой о малых деформациях.

Received August 08, 2007